A homogeneous model for compressible and immiscible two-phase flows: 2. Properties of the model

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We consider the set of PDE in a one-dimensionnal framework:

$$\begin{cases} \dfrac{\partial}{\partial t}(\rho \underline{Y}) + \dfrac{\partial}{\partial x}(\rho U \underline{Y}) = \rho \dfrac{\overline{\underline{Y}} - \underline{Y}}{\lambda}, \\ \dfrac{\partial}{\partial t}(\rho) + \dfrac{\partial}{\partial x}(\rho U) = 0, \\ \dfrac{\partial}{\partial t}(\rho U) + \dfrac{\partial}{\partial x}\left(\rho U^2 + P\right) = 0, \\ \dfrac{\partial}{\partial t}(\rho E) + \dfrac{\partial}{\partial x}(U(\rho E + P)) = 0, \end{cases}$$

with the following definitions,

- the total specific energy of the mixture E is the sum of the specific internal-energy e and the specific kinetic-energy: $E = e + U^2/2$;
- the fractions are:

  $$\underline{Y} = (\alpha_1, y_1, z_1), \quad \alpha_2 = 1 - \alpha_1, \quad y_2 = 1 - y_1, \quad z_2 = 1 - z_1,$$

  and the equilibrium fractions are:

  $$\overline{\underline{Y}} = (\overline{\alpha_1}, \overline{y_1}, \overline{z_1}), \quad \overline{\alpha_2} = 1 - \overline{\alpha_1}, \quad \overline{y_2} = 1 - \overline{y_1}, \quad \overline{z_2} = 1 - \overline{z_1},$$

  We recall that $\overline{\underline{Y}}$ is defined as the fraction such that the mixture entropy is maximum at a given e and τ, hence **$\overline{\underline{Y}}$ only depends on** (τ, e).

- and the EOS (pressure law) for the mixture:

$$\frac{P(\underline{Y},\tau,e)}{T(\underline{Y},\tau,e)} = \left(\alpha_1 \frac{P_1}{T_1} + \alpha_2 \frac{P_2}{T_2}\right) \quad \text{and} \quad \frac{1}{T(\underline{Y},\tau,e)} = \frac{z_1}{T_1} + \frac{z_2}{T_2},$$

where the phasic temperature-laws and the phasic pressure-laws are obtained from the phasic entropies (given by the user):

$$T_k^{-1} = T_k^{-1}(\tau_k, e_k) = \frac{\partial s_k}{\partial e_k}_{|\tau_k}(\tau_k, e_k)$$

$$P_k = P_k(\tau_k, e_k) = T_k(\tau_k, e_k)\,\frac{\partial s_k}{\partial \tau_k}_{|e_k}(\tau_k, e_k)$$

We also have $e_k = \dfrac{z_k e}{y_k}$ and $\tau_k = \dfrac{\alpha_k \tau}{y_k}$.

Finally, provided that $\lambda(\underline{Y},\tau,e,U)$ is given, the system is closed. The six PDE's define the time and space evolution of the set of six variables $(\alpha_1, y_1, z_1, \tau, e, U)$

The model is in conservative form, it involves first order derivatives in time and space and a source term.

The main properties of this model are the following.

- The model is based on the Euler set of equations and thus inherits from its wave structure.
- If the specific phasic entropies are strictly concave and if $T_k > 0$, the model is hyperbolic.
- Shocks are defined through Rankine-Hugoniot jump relations.
- We have a positivity result for the fractions.

The following results are presented in a "weakly rigorous" manner, but more mathematicaly rigorous definitions and proofs can be found in many books (for instance [Godlewski-Raviart, 1996] or [Smoller,])

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We focuse here on the convective part of the system. The source terms are omitted:

$$\begin{cases} \dfrac{\partial}{\partial t}(\rho \underline{Y}) + \dfrac{\partial}{\partial x}(\rho U \underline{Y}) = 0, \\ \dfrac{\partial}{\partial t}(\rho) + \dfrac{\partial}{\partial x}(\rho U) = 0, \\ \dfrac{\partial}{\partial t}(\rho U) + \dfrac{\partial}{\partial x}\left(\rho U^2 + P\right) = 0, \\ \dfrac{\partial}{\partial t}(\rho E) + \dfrac{\partial}{\partial x}(U(\rho E + P)) = 0. \end{cases}$$

This model can be written in the conservative form:

$$\frac{\partial}{\partial t}(W) + \frac{\partial}{\partial x}(F(W)) = 0,$$

or in equivalent manner for regular solutions, in the non-conservative form:

$$\frac{\partial}{\partial t}(W) + A(W)\frac{\partial}{\partial x}(W) = 0, \quad \text{with} \quad A(W) = \nabla_W(F(W)).$$

In order to study the wave structure of this set of first-order PDE, we study the eigenstructure of the jacobian matrix $A(W)$.

Definition of hyperbolicity: *The system is hyperbolic when the matrix A has n real eigenvalues and when its set of right eigenvectors spans $\mathbb{R}^n$.*

Annexe $\Longrightarrow$ ...

Remark. *The Jacobian matrix can be calculated for any set of variables $Z = \Phi(W)$ provided that Φ is a diffeomorphism ($W = \Phi^{-1}(Z)$). Then we have:*

$$\frac{\partial}{\partial t}(Z) + B(Z)\frac{\partial}{\partial x}(Z) = 0,$$

with

$$B(Z) = \left(\nabla_Z\left(\Phi^{-1}(Z)\right)\right)^{-1} A\left(\Phi^{-1}(Z)\right)\nabla_Z\left(\Phi^{-1}(Z)\right).$$

The diffeomorphism Φ does not change the eigenvalues, but the eigenvectors are transformed according to Φ.

The Jacobian matrix $A(\underline{Y}, s, U, P)$ of the system is:

$$A(\underline{Y}, s, U, P) = \begin{pmatrix} U\,\mathbb{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U & 0 & 0 \\ \mathbf{0} & 0 & U & \tau \\ \mathbf{0} & 0 & C^2/\tau & U \end{pmatrix},$$

where C is the sound speed of the mixture:

$$C^2 = \tau^2 \left(P \frac{\partial P}{\partial e}_{|\underline{Y},\tau} - \frac{\partial P}{\partial \tau}_{|\underline{Y},e} \right) \qquad \left(= \frac{\partial P}{\partial \rho}_{|\underline{Y},s} \right).$$

The eigenvalues are easy to find: $\lambda_1 = U - C$, $\lambda_{2,3,4,5} = U$ and $\lambda_6 = U + C$, and the associated eigenvectors are:

$$\begin{aligned} r_2 &= (1,0,0,0,0,0)^\top \\ r_3 &= (0,1,0,0,0,0)^\top \\ r_4 &= (0,0,1,0,0,0)^\top \\ r_5 &= (0,0,0,1,0,0)^\top \end{aligned} \qquad \begin{aligned} r_1 &= (0,0,0,0,\tau,-C)^\top \\ r_6 &= (0,0,0,0,\tau,C)^\top \end{aligned}$$

Hence, the model is hyperbolic if and only if $C^2 > 0$.

Using the mixture pressure-law, the sound speed C can also be written:

$$\begin{aligned} \frac{C^2}{T\tau^2} = & \quad \frac{1}{y}(-\alpha, Pz).(-s_1'').\begin{pmatrix} -\alpha \\ Pz \end{pmatrix} \\ + & \quad \frac{1}{1-y}(-(1-\alpha), P(1-z)).(-s_2'').\begin{pmatrix} -(1-\alpha) \\ P(1-z) \end{pmatrix} \end{aligned} \qquad (1)$$

where $(\tau_k, e_k) \to s_k''(\tau_k, e_k)$ stands for the Hessian matrix of the specific entropies s_k. We recall that the phasic sound speed of the phase k is defined as:

$$\frac{c_k^2}{T_k \tau_k^2} = (-1, P_k).(-s_k'').(-1, P_k)^\top \qquad (2)$$

Hence, if the specific entropies are strictly concave and if they ensure that the phasic temperature T_k are non-negative, then $C^2 > 0$ and the model is hyperbolic.

Additionnal remarks

Using the variables $Z = (\underline{Y}, s, U, P)$, we have:

$$R\frac{\partial}{\partial t}(Z) + RA\ \frac{\partial}{\partial x}(Z) = 0,$$

with,

$$R = \begin{pmatrix} \mathbb{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 & 0 \\ \mathbf{0} & 0 & (\rho C)^2 & 0 \\ \mathbf{0} & 0 & 0 & 1 \end{pmatrix} \text{ and } RA = \begin{pmatrix} U\ \mathbb{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U & 0 & 0 \\ \mathbf{0} & 0 & U(\rho C)^2 & \rho C^2 \\ \mathbf{0} & 0 & \rho C^2 & U \end{pmatrix}$$

The matrix R is symmetric positive definite if $C \neq 0$ and RA is symmetric. **The system is thus symmetrizable provided that** $C \neq 0$.

As a consequence, there exist a local-in-time smooth solution to the Cauchy problem (Kato's theorem).

If the system is symmetrizable, it is hyperbolic (a $n \times n$ symmetric matrix has real eigenvalues and its set of right eigenvectors spans $\mathbb{R}^n$).

Additionnal remarks

It can be shown that for our model:

- $\lambda_{2,3,4,5} = U$ are contact discontinuity waves (linearly degenerated fields);
- $\lambda_1 = U - C$ and $\lambda_6 = U + C$ are shock/rarefaction waves (genuinely non-linear waves).

- The rarefaction waves are regular parts of the solutions.
- The shock waves and the contact discontinuities are associated with discontinuities in the solutions.

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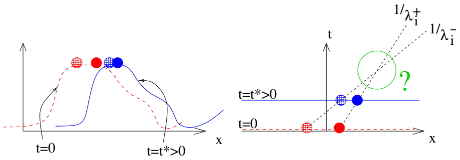
In the previous section, we have considered smooth solutions, but it has been mentionned that discontinuities may appear when the solution is compressed:



These patterns correspond to shock waves and they have to be defined.

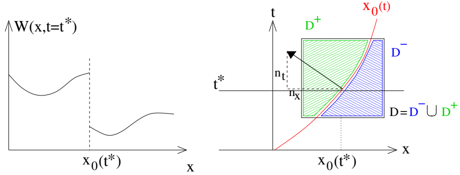
Since we are dealing with discontinuous solutions, we consider the system in conservative form:

$$\frac{\partial}{\partial t}(W)+\frac{\partial}{\partial x}(F(W))=0,$$

and the latter should be understood in a weak sense. That is for any test function ϕ with a compact support:

$$\int_{\mathbb{R}^6\times(0,\infty)}\left(W\frac{\partial}{\partial t}(\phi)+F(W)\frac{\partial}{\partial x}(\phi)\right)dxdt=0,$$

Let us assume that the solution W is smooth except at a point $x_0(t)$:

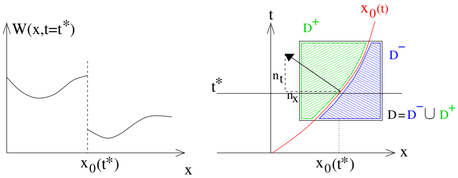


and let us integrate over $D = D^- \cup D^+$ with ϕ a test function with a compact support in D:

$$\int_D \left(W \frac{\partial}{\partial t}(\phi) + F(W) \frac{\partial}{\partial x}(\phi) \right) dxdt = 0.$$

We then have:

$$\int_D \left(W \frac{\partial}{\partial t}(\phi) + F(W) \frac{\partial}{\partial x}(\phi) \right) dxdt = 0,$$
$$\int_{D^-} \left(W \frac{\partial}{\partial t}(\phi) + F(W) \frac{\partial}{\partial x}(\phi) \right) dxdt + \int_{D^+} \left(W \frac{\partial}{\partial t}(\phi) + F(W) \frac{\partial}{\partial x}(\phi) \right) dxdt = 0,$$



and using the Green's formula:

$$\int_{x_0(t)} \left(n_t W_{|D^-} + n_x F(W_{|D^-}) \right) \phi \, d\xi - \int_{D^-} \left(\phi \frac{\partial}{\partial t}(W) + \phi \frac{\partial}{\partial x}(F(W)) \right) dxdt$$
$$- \int_{x_0(t)} \left(n_t W_{|D^+} + n_x F(W_{|D^+}) \right) \phi \, d\xi - \int_{D^+} \left(\phi \frac{\partial}{\partial t}(W) + \phi \frac{\partial}{\partial x}(F(W)) \right) dxdt = 0$$

On the interior of D^- and D^+, W is smooth and is a solution in the strong sense, hence the two terms on the right vanishe. We then obtain:

$$\int_{x_0(t)} \left(n_t \left(W_{|D^+} - W_{|D^-} \right) + n_x \left(F(W_{|D^+}) - F(W_{|D^-}) \right) \right) \phi \, d\xi = 0.$$

This relation holds for any ϕ, hence we get:

$$-\sigma\left(W^+ - W^-\right) + \left(F(W^+) - F(W^-)\right) = 0$$

with $\sigma = -n_t/n_x$ ($n_x \neq 0$, i.e. finite speed of propagation) the velocity of the discontinuity in direction x.

This relation is called the Rankine-Hugoniot jump relation, and we use the Classical notation:

$$-\sigma[W] + [F(W)] = 0$$

Consequence: a smooth solution on D^- and a smooth solution on D^+ form a weak solution on $D = D^- \cup D^+$ if and only if there exist σ such that $-\sigma[W] + [F(W)] = 0$ (caution : σ is a scalar and it is the same for all the components of W).

In the following, we assume that the state W^- is known, and we want to find a state W^+ that fulfills the RH jump relations.

The RH jump relations for our model are:

$$\begin{cases} -\sigma[\rho \underline{Y}] + [\rho U \underline{Y}] = 0, \\ -\sigma[\rho] + [\rho U] = 0, \\ -\sigma[\rho U] + [\rho U^2 + P] = 0, \\ -\sigma[\rho E] + [U(\rho E + P)] = 0. \end{cases}$$

We define $J = \rho(U - \sigma)$. For any quantity a, we define $\bar{a} = (a^+ + a^-)/2$ and we will use the relation:

$$[ab] = \bar{a}[b] + [a]\bar{b}.$$

Since $[\sigma] = 0$, we can rewrite the jump relations as:

$$\begin{cases} J[\underline{Y}] = 0, \\ [J] = 0, \\ J[U] + [P] = 0, \\ J[e + U^2/2] + [UP] = 0, \end{cases} \quad \begin{cases} J[\underline{Y}] = 0, \\ [J] = 0, \\ J[U] + [P] = 0, \\ J[e] + \bar{P}[U] = 0, \end{cases} \quad \begin{cases} J[\underline{Y}] = 0, \\ [J] = 0, \\ J^2[\tau] + [P] = 0, \\ J([e] + \bar{P}[\tau]) = 0, \end{cases}$$

where we have used the relation: $J = \rho(U - \sigma) \Rightarrow \tau J = U - \sigma \Rightarrow J[\tau] = [U]$.

Contact discontinuity

If we have $J = 0$, we get:

$$\rho^+(U^+ - \sigma) = 0 \quad \text{and} \quad \rho^-(U^- - \sigma) = 0,$$

which, for $\rho^+ > 0$ and $\rho^- > 0$, gives:

$$U^+ = U^- = \sigma.$$

This corresponds to a **contact discontinuity**. The RH jump relations are then:

$$\begin{cases} [P] = 0 \\ [U] = 0 \end{cases}$$

We have a system of 2 non-linear equations for 6 variables: $\underline{Y}^+$, τ^+, U^+, e^+.

The velocity is $U^+ = U^-$ and it remains to find $\underline{Y}^+$, τ^+ and e^+ such that:

$$[P] = 0 \Leftrightarrow P(\underline{Y}^+, \tau^+, e^+) - P(\underline{Y}^-, \tau^-, e^-) = 0.$$

A contact discontinuity contains a lot of degrees of freedom.

Shock wave

We focus now on the case $J \neq 0$. We choose a value for J, the RH jump relations can be written:

$$\left\{\begin{array}{l} J[\underline{Y}]=0, \\ J^2[\tau]+[P]=0, \\ J([e]+\bar{P}[\tau])=0, \\ [J]=0, \end{array}\right. \quad \left\{\begin{array}{llll} [\underline{Y}]=0, & \longrightarrow & \text{gives} & \underline{Y}^+, \\ J^2[\tau]+[P]=0, & \longrightarrow & \text{gives} & (\tau^+,e^+), \\ [e]+\bar{P}[\tau]=0, & \longrightarrow & \text{gives} & (\tau^+,e^+), \\ [U]=J[\tau], & \longrightarrow & \text{gives} & U^+. \end{array}\right.$$

We then have a system of 6 non-linear equations and 6 unknown: $\underline{Y}^+$, τ^+, U^+, e^+, with a 2×2 subsystem that defines (τ^+,e^+). The velocity of the shock is then given by: $\sigma = U^+ - J\tau^+ = U^- - J\tau^-$.

Remark.

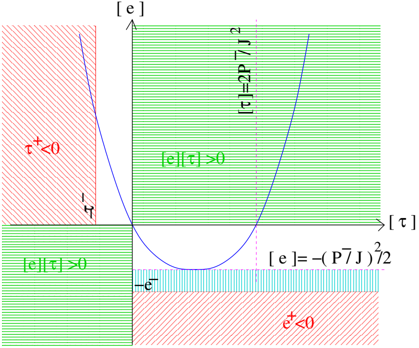
- *If $J=\rho(U-\sigma)>0$, we have $U^+>\sigma$ and $U^->\sigma$. Hence we are considering a $U-C$ shock wave.*
- *If $J=\rho(U-\sigma)<0$, we have $U^+<\sigma$ and $U^-<\sigma$. Hence we are considering a $U+C$ shock wave.*

Shock wave

Then to define the shock, the main problem lies in the subsytem for (τ^+, e^+):

$$\left\{ \begin{array}{l} J^2[\tau] + [P] = 0, \\ [e] + \bar{P}[\tau] = 0. \end{array} \right. \rightarrow \left\{ \begin{array}{l} J^2[\tau] + (P^+ - P^-) = 0, \\ [e] + \frac{P^+ + P^-}{2}[\tau] = 0. \end{array} \right.$$

From the two jump relations above, one can write: $[e] = [\tau]\left(\frac{J^2}{2}[\tau] - P^-\right)$.



- It can be shown from the RH relations that $[e][\tau] \leq 0$, the green zone is not allowed.
- e^+ and τ^+ should be non-negative, the red zones is not allowed.
- For $[e] < \frac{1}{2}\left(\frac{P^-}{J}\right)^2$, there is no solution. The cyan zone is not allowed (the cyan zone can be included in the zone $e^+ < 0$).

Shock wave

The subsytem for (τ^+, e^+) can also be written:

$$\left\{ \begin{array}{l} J^2[\tau]+[P]=0, \\ [e]+\bar{P}[\tau]=0, \end{array} \right. \rightarrow \left\{ \begin{array}{l} J^2[\tau]+[P]=0, \\ [e]=[\tau]\left(\frac{J^2}{2}[\tau]-P^-\right), \end{array} \right.$$

or:

$$\left\{ \begin{array}{l} J^2(\tau^+-\tau^-)+(P(\underline{Y},\tau^+,\varepsilon(\tau^+))-P^-)=0, \\ \varepsilon(\tau^+)=e^-+(\tau^+-\tau^-)\left(\frac{J^2}{2}(\tau^+-\tau^-)-P^-\right). \end{array} \right.$$

The solution is then given by solving the non-linear equation for τ^+: $\mathscr{F}(\tau^+)=0$, where we have

$$\mathscr{F}(X)=J^2(X-\tau^-)+(P(\underline{Y},X,\varepsilon(X))-P^-).$$

This equation may have several solutions depending on the pressure law. **We have to had a criterion to select an unique solution**.

Shock wave

Several criterions have been proposed in the literature: Liu, Lax, vanishing viscosity/entropy dissipation ...

These criterions are sometimes (depending on the system) equivalent, but not always.

The "entropy" criterion:

- In thermodynamics, "slow" transformations are associated with **reversible processes, so that the entropy is constant:** $ds = 0$ (it has not been discussed here, but rarefaction waves are such that $ds = 0$).
- A shock is a "fast" transformation and it can thus be considered as an **irreversible process, so that the entropy should increase** $ds \geq 0$. This yields in a weak sense :

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}(\rho U s) \geq 0$$

or considering the jump relation:

$$-\sigma[\rho s] + [\rho U s] \geq 0 \iff J[s] \geq 0 \iff J(s(\underline{Y}, \tau^+, \varepsilon(\tau^+)) - s^-) \geq 0.$$

Plan

We focus here on an important property for the present model: under classical assumptions, the frations remain positive.

We will use the following lemma.

Let Ω be a subset of $\mathbb{R}$. Let Ψ, a, Π and U be some sufficiently regular applications on $\Omega \times [0, T]$, with the following properties:

- $\Psi : (x,t) \mapsto \Psi(x,t)$;
- $a : (x,t) \mapsto a(x,t)$, $a \in L^\infty$;
- $\Pi : (x,t) \mapsto \Pi(x,t)$, $\Pi \geq 0$;
- $U : (x,t) \mapsto U(x,t)$, $U \in L^\infty$ and $\dfrac{\partial}{\partial x}(U) \in L^\infty$;

and such that:

$$\frac{\partial}{\partial t}(\Psi(x,t)) + U \frac{\partial}{\partial x}(\Psi(x,t)) = a(x,t)\Psi(x,t) + \Pi(x,t).$$

Suppose that for all $x_b \in \partial\Omega$, the boundary of Ω, $\Psi(x_b,t) \geq 0$ if $(U \cdot n)(x_b,t) \leq 0$, where n is the outward normal of Ω. With all these assumptions, if $\Psi(x,t=0) \geq 0$ then for all $0 \leq t \leq T$, $\Psi(x,t) \geq 0$.

We have for the volume fraction α_i:

$$\frac{\partial}{\partial t}(\alpha_i(x,t)) + U\frac{\partial}{\partial x}(\alpha_i(x,t)) = a_i(x,t)\alpha_i(x,t) + \Pi_i(x,t),$$

with

$$a_i = -1/\lambda \quad \text{and} \quad \Pi_i = \overline{\alpha}_i/\lambda.$$

Since the equilibrium volume fraction $\overline{\alpha}_i$ belongs to $[0,1]$ and the time scale λ must be chosen non-negative, we obviously have

$$\Pi_i \geq 0.$$

Hence if we assume that the solution is regular (no shock), the lemma can be straightforward applied for $\Psi = \alpha_i$, $i = 1,2$.

We thus have $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, and since $\alpha_1 + \alpha_2 = 1$ we get that:

$$\alpha_i \in [0,1].$$

Obviously, we can also prove that:

$$y_i \in [0,1] \quad \text{and} \quad z_i \in [0,1].$$

In the case of discontinuous solutions, the lemma can not be applied.

- For contact discontinuities, we have (previous section) to choose $[\underline{Y}]$. Hence the contact discontinuity does not modify the positivity of the fraction.
- For shock waves, U does not belong to L^∞. The lemma can not be applied. Nevertheless, we have (previous section): $[\underline{Y}] = 0$. So that there is no loss of posivity through a shock wave.

Finally, we can state that the model is such that the fractions α_i, y_i and z_i remain in $[0, 1]$.

Plan

We consider the set of first-order PDE's in the following non-conservative form:

$$\frac{\partial}{\partial t}(W)+A(W)\frac{\partial}{\partial x}(W)=0,$$

where W is a vector with dimension n and A is a $n\times n$-matrix.

Definition of hyperbolicity: *The system is hyperbolic when the matrix A has n real eigenvalues and when its set of right eigenvectors spans $\mathbb{R}^n$.*

Illustration of hyperbolicity on the linear case: we assume now that the matrix A does not depend on W, x or t

A monochromatic wave $\underline{W} = W_0 e^{(i\omega t - ikx)}$ is a solution of the system of

$$\frac{\partial}{\partial t}(W) + A\frac{\partial}{\partial x}(W) = 0,$$

if and only if: $\underline{W}(Ak - \omega I_n) = 0$.

We set λ a complex eigenvelue of A: $\lambda = \lambda_a + i\lambda_b$. We then choose $k_0 > 0$ and we also choose ω_0 such that $\omega_0 = k_0\lambda$.

Then, the monochromatic wave associated with k_0 and ω_0 is solution of the system of PDE's if and only if:

$$e^{(i\omega_0 t - ik_0 x)} = e^{(i\lambda k_0 t - ik_0 x)} = e^{(i\lambda_a k_0 t - ik_0 x)} e^{-\lambda_b k_0 t}$$

It appears that the solution is stable in time iff $\lambda_b \geq 0$.

But when dealing with physical phenomena, the matrix A often (always ?) contains real coefficients. Hence, if A has one complex eigenvalue, its conjugate is also an eigenvalue. This implies that when A has complex eigenvalues, there is always at least one of these that leads to unstable monochromatic waves.

Unfortunately, in practice we are dealing with solutions that contain a wide range of frequencies. Moreover, the phenomena of interest are often associated with time/space-scale that are related to the eigenvalues of A.

Hence, it is necessary (but not sufficient) to work with models whose Jacobian matrix have real eigenvalues and a complete set of right eigenvectors.

If the right eigenvectors V_i $(i=1..n)$ **of** A **span** $\mathbb{R}^n$, the matrix $\mathscr{P}$ whose columns are the eigenvectors of A is such that:

$$A=\mathscr{P}D\mathscr{P}^{-1},$$

where D is a diagonal matrix that contains the eigenvalues λ_i of A, with $AV_i=\lambda_i V_i$.

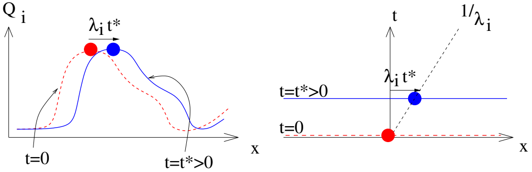
By multipling the system by $\mathscr{P}^{-1}$ we get:

$$\mathscr{P}^{-1}\left(\frac{\partial}{\partial t}(W)+A\frac{\partial}{\partial x}(W)\right)=0,$$
$$\frac{\partial}{\partial t}\left(\mathscr{P}^{-1}W\right)+\mathscr{P}^{-1}A\mathscr{P}\frac{\partial}{\partial x}\left(\mathscr{P}^{-1}W\right)=0,$$
$$\frac{\partial}{\partial t}(Q)+D\frac{\partial}{\partial x}(Q)=0,\quad \text{with} \quad Q=\mathscr{P}^{-1}W.$$

The projection on the basis of the eigenvectors allows to uncouple the n PDE of the system. Since D is diagonal, each component i of the vector Q is then advected with the velocity λ_i,

$$\frac{\partial}{\partial t}(Q_i) + \lambda_i \frac{\partial}{\partial x}(Q_i) = 0,$$

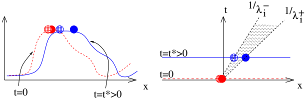
and the solution is: $Q_i(x,t) = Q_i(x - \lambda_i t, t = 0)$.



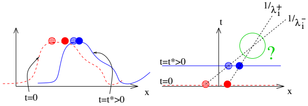
Finally, the solution for W is $W(x,t) = \mathscr{P} Q(x,t)$.

The non-linear case is obviously not so easy. If A depends on W, the eigenvalues also depend on W. The initial pattern ($t = 0$) is not simply translated in the plane (x, t), it may also be deformed ! Three situations may be encountered.

- The initial pattern is stretched. It corresponds to a rarefaction wave.



- The initial pattern is compressed. Shocks may appear. A shock is a discontinuity of the solution.



- The initial pattern is translated as in the linear case = contact discontinuity.