CFL-less explicit schemes for conservation laws based on a kinetic approach

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Outlines

Motivation: kinetic approximations of conservation laws allow to design explicit CFL-less high order schemes. But they involve hidden variables. How to apply boundary conditions on these variables ?

Kinetic relaxation and over-relaxation

Relaxation of hyperbolic systems

- Hyperbolic system with unknown $\mathbf{u}(x,t) \in \mathbb{R}^m$:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0.$$

LHS: non-linear equations ☹; RHS: zero ☺.

- Approximation by Jin-Xin[1] relaxation ($\lambda > 0$, $\varepsilon \to 0^+$)

$$\partial_t \mathbf{u} + \partial_x \mathbf{z} = \mathbf{0}, \tag{1}$$

$$\partial_t \mathbf{z} + \lambda^2 \partial_x \mathbf{u} = \mu, \tag{2}$$

where

$$\mu = \frac{1}{\varepsilon}(\mathbf{f}(\mathbf{u}) - \mathbf{z}).$$

LHS: linear system with constant coefficients ☺; RHS: non-linear coupling ☹.

[1] Jin and Xin, "The relaxation schemes for systems of conservation laws in arbitrary space dimensions".

Over-relaxation

Let's do splitting. For a rigorous formulation, introduce the Dirac comb:

$$\Psi(t) = \sum_{i\in\mathbb{Z}} \delta(t - i\Delta t).$$

Jin-Xin relaxation is replaced in practice by

$$\partial_t \mathbf{u} + \partial_x \mathbf{z} = 0, \tag{3}$$

$$\partial_t \mathbf{z} + \lambda^2 \partial_x \mathbf{u} = \mu, \tag{4}$$

with

$$\mu(x,t) = \theta\Psi(t)\left(\mathbf{f}(\mathbf{u}(x,t)) - \mathbf{z}(x,t^-)\right), \quad \theta \in [1,2].$$

In other words, at times $t = i\Delta t$, $\mathbf{z}$ has jumps in time and:

$$\mathbf{z}(x,t^+) = \theta\mathbf{f}(\mathbf{u}(x,t)) + (1-\theta)\mathbf{z}(x,t^-).$$

If the relaxation parameter $\theta = 1$, we recover the first order splitting.
The **over-relaxation** corresponds to $\theta = 2$.

explicit, CFL-less Kinetic interpretation

We can diagonalize the linear hyperbolic operator. For this, consider the change of variables

$$\mathbf{k}^+ = \frac{\mathbf{u}}{2} + \frac{\mathbf{z}}{2\lambda}, \quad \mathbf{k}^- = \frac{\mathbf{u}}{2} - \frac{\mathbf{z}}{2\lambda}.$$

$$\mathbf{u} = \mathbf{k}^+ + \mathbf{k}^-, \quad \mathbf{z} = \lambda \mathbf{k}^+ - \lambda \mathbf{k}^-.$$

Then

$$\partial_t \mathbf{k}^+ + \lambda \partial_x \mathbf{k}^+ = \mathbf{r}^+, \quad \partial_t \mathbf{k}^- - \lambda \partial_x \mathbf{k}^- = \mathbf{r}^-,$$

where

$$\mathbf{r}^\pm(x,t) = \theta \Psi(t) \left(\mathbf{k}^{eq,\pm}(\mathbf{u}(x,t^-)) - \mathbf{k}^\pm(x,t^-) \right)$$

and the "Maxwellian" states $\mathbf{k}^{eq,\pm}$ are given by

$$\mathbf{k}^{eq,\pm}(\mathbf{u}) = \frac{\mathbf{u}}{2} \pm \frac{\mathbf{f(u)}}{2\lambda}.$$

Most of the time, the kinetic variables $\mathbf{k}^+$ and $\mathbf{k}^-$ satisfy free transport equations at velocity $\pm\lambda$, with relaxation to equilibrium at each time step.

Equivalent PDE and boundary conditions

Oscillations of the flux error

We consider the case $\theta = 2$.

- Let us introduce the "flux error"

$$\mathbf{y} := \mathbf{z} - \mathbf{f}(\mathbf{u}).$$

- At time $t = i\Delta t$, we see that

$$\mathbf{y}(x, t^+) = -\mathbf{y}(x, t^-).$$

  Therefore $\mathbf{y}$ oscillates around 0 at a frequency $1/\Delta t$.

- For the analysis, it is better to consider the solution only at even (or only at odd) times steps $t = 2i\Delta t$.

Equivalent PDE analysis

We can prove the following result (more rigorous formulation exists[2]).
Theorem: if the solution of the over-relaxation scheme is considered at even time steps, then, up to second order terms in Δt, its equivalent equation in $(\mathbf{u}, \mathbf{y})$ is the following hyperbolic system of conservation laws

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0,$$
$$\partial_t \mathbf{y} - \mathbf{f}'(\mathbf{u}) \partial_x \mathbf{y} = 0.$$

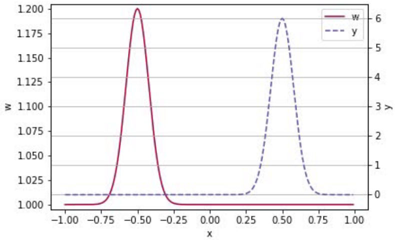
Remarks:

- $\mathbf{u}$ satisfies the expected conservative system at order $O(\Delta t^2)$.
- $\mathbf{y}$ satisfies a non-conservative equation.
- There is no assumption on the smallness of $\mathbf{y}$ at the initial time.
- The waves for $\mathbf{u}$ and $\mathbf{y}$ move in opposite directions.

[2] Drui et al., "An analysis of over-relaxation in a kinetic approximation of systems of conservation laws".

Numerical results

- Isothermal Euler equations
$$\mathbf{u} = (\rho, \rho u)^T, \quad \mathbf{f}(\mathbf{u}) = (\rho u, \rho u^2 + c^2 \rho).$$
- Smooth initial data with a bump. Supersonic flow moving rightward ($0 < \lambda_1 = u - c < \lambda_2 = u + c$). Non-physical initial value of $\mathbf{y} \neq 0$.
- Transport equations solved with an exact characteristic scheme (Lattice-Boltzmann Method).
- We plot ρ and the first component of $\mathbf{y}$ at even time steps. We clearly observe the opposite propagation of the waves.



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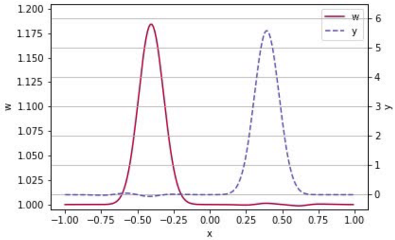

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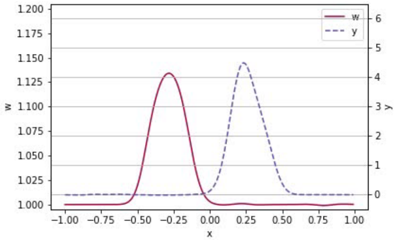

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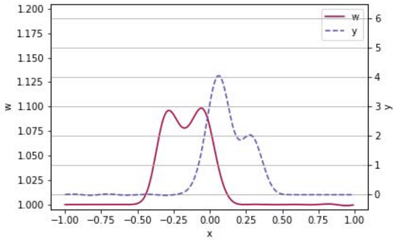

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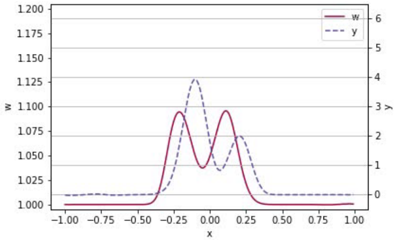

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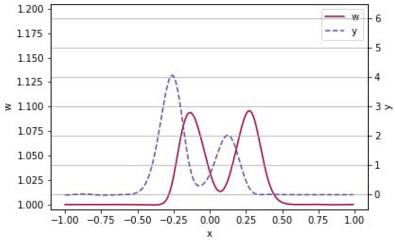

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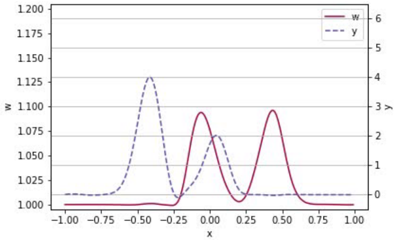

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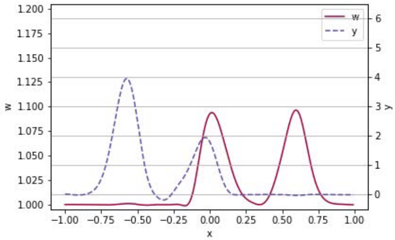

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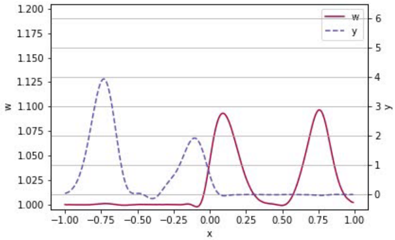


Application to boundary conditions

The analysis gives hints to build stable boundary conditions on $\mathbf{z}$.

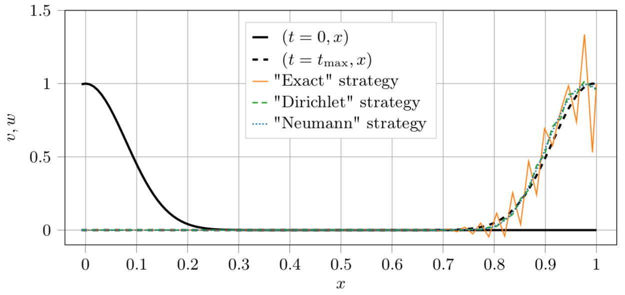
- Roughly speaking, at an inflow boundary for $\mathbf{u}$, one should impose $\mathbf{u}$ and not $\mathbf{y}$, while at an outflow boundary for $\mathbf{u}$ one should impose $\mathbf{y}$ and not $\mathbf{u}$.
- This also means that one should impose exactly m boundary conditions, where m is the dimension of $\mathbf{u}$. This is compatible with the characteristics of the kinetic system.
- We expect that $\mathbf{y} \simeq 0$. However it is better to impose a Neumann boundary condition $\partial_x \mathbf{y} = 0$ for not perturbing the time oscillations of $\mathbf{y}$.

Numerical results

We consider a simple transport equation at constant velocity $v > 0$.

$$\mathbf{u} = \rho, \quad \mathbf{f}(\mathbf{u}) = \rho v.$$

Smooth initial data. Three strategies of boundary conditions.
Typical plot for the three stategies.

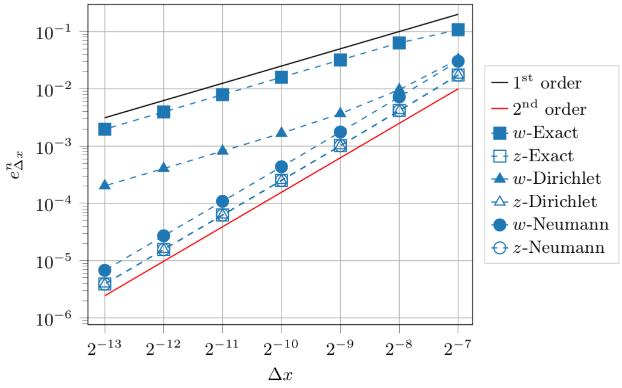


Error rate

Isothermal Euler equations

$$\mathbf{u} = (\rho, \rho u)^T, \quad \mathbf{f}(\mathbf{u}) = (\rho u, \rho u^2 + c^2 \rho).$$

Smooth initial data corresponding to a supersonic flow moving rightward.
We test the previous three strategies of boundary conditions.
Error rate for the three strategies



Kinetic relaxation in higher dimensions

Kinetic model in higher dimensions[3,4]

- Vectorial kinetic equation

$$\partial_t \mathbf{k} + \sum_{i=1}^{D} \mathbf{V}^i \partial_i \mathbf{k} = \frac{1}{\tau}(\mathbf{k}^{eq}(\mathbf{k}) - \mathbf{k}). \tag{5}$$

  $\mathbf{k}(\mathbf{x}, t) \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^D$.

- The matrices $\mathbf{V}^i$, $1 \leq i \leq D$ are **diagonal** and **constant**.
- $\mathbf{u} = \mathbf{P}\mathbf{k}$ where $\mathbf{P}$ is a constant $m \times n$ matrix, $m < n$.
- The equilibrium distribution $\mathbf{k}^{eq}(\mathbf{k})$ is such that $\mathbf{P}\mathbf{k} = \mathbf{P}\mathbf{k}^{eq}(\mathbf{k})$.
- When $\tau \to 0$, approximation of $\partial_t \mathbf{u} + \sum_{i=1}^{D} \partial_i \mathbf{f}^i(\mathbf{u}) = 0$, where the flux is given by $\mathbf{f}^i(\mathbf{u}) = \mathbf{P}\mathbf{V}^i\mathbf{k}^{eq}(\mathbf{k})$.

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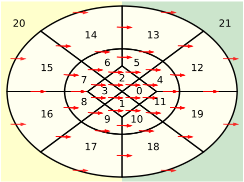
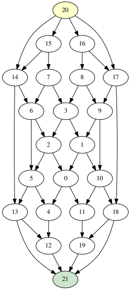
[3] Bouchut, "Construction of BGK models with a family of kinetic entropies for a given system of conservation laws".

[4] Aregba-Driollet and Natalini, "Discrete kinetic schemes for multidimensional systems of conservation laws".

CFL-less kinetic DG scheme

- On unstructured meshes, it is easy to solve the kinetic transport equations with an implicit upwind Discontinuous Galerkin scheme.
- In practice, the scheme is **explicit** if the cells are visited in the good order.





- In this way we obtain **explicit unconditionnaly stable** schemes !

CFL-less kinetic DG scheme

Further improvements

- High order in space and time with palindromic splitting[5];
- Easy parallelization, with a task-based approach and StarPU runtime system[6];
- Applications to: compressible flows, MHD, two-phase flow, *etc.*[7]

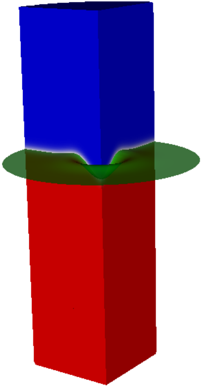
[5] Hairer, Lubich, and Wanner, *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*.

[6] Badwaik et al., "Task-based parallelization of an implicit kinetic scheme".

[7] Coulette et al., "High-order implicit palindromic Discontinuous Galerkin method for kinetic-relaxation approximation".

Thanks for your attention !

Rayleigh-Taylor instability. Two immiscible fluids with gravity. CFL=10.



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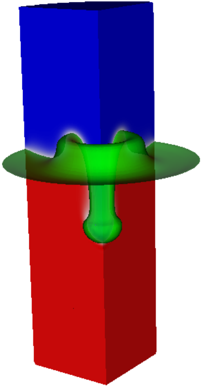

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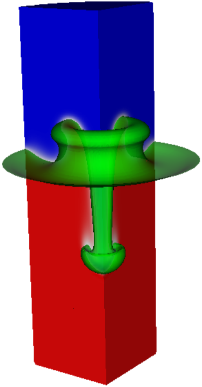

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